$$\begin{array}{ccccl}\tilde{C} & \supset & \tilde{\mathcal{U}} & = & \tilde{C} \setminus \bigcup_{\alpha \in \Phi} \tilde{C}^{\alpha} \\ {\scriptstyle \pi}\downarrow & & \downarrow & & \\ C & \supset & \mathcal{U} & & \end{array}$$

$G \rightsquigarrow$ root datum $(X_*, X^*, \check{\Phi}, \Phi)$ over $k = \bar{k}$, $\mathrm{char}(k) \nmid |W|$

Jac = Jacobian of $\tilde{C}$

$= H^1(\tilde{C}, \mathbb{G}_m)^0$

$\mathcal{L}$ Poincaré line bdle on $\mathrm{Jac} \times \mathrm{Jac}$ with $(w,w)^* \mathcal{L} \simeq \mathcal{L}$ $\forall w \in W$

$\mathrm{Jac} \otimes X_*$, $\mathrm{Jac} \otimes X^*$ are dual abelian varieties:

$X_* = \bigoplus_i \mathbb{Z} e_i$

$X^* = \bigoplus_i \mathbb{Z} f_i$

dual bases, with $\tilde{pr}_i : \mathrm{Jac} \otimes X^* \longrightarrow \mathrm{Jac}$, $\iota_i : \mathrm{Jac} \longrightarrow \mathrm{Jac} \otimes X_*$ } attached to e_i

$pr_i : \mathrm{Jac} \otimes X_* \longrightarrow \mathrm{Jac}$, $\tilde{\iota}_i : \mathrm{Jac} \longrightarrow \mathrm{Jac} \otimes X^*$ } attached to f_i

then $\mathcal{L}_T := \bigotimes_{i=1}^{n} (pr_i \times \tilde{pr}_i)^* \mathcal{L}$ is a dualizing line bundle $\square$

on $(\mathrm{Jac} \otimes X_*) \times (\mathrm{Jac} \otimes X^*)$, and W-invariant.

$A := (\mathrm{Jac} \otimes X^*)_W$ coinvariant abelian variety,

quotient by $\sum_{w \in W} (1-w)(\mathrm{Jac} \otimes X^*)$

$\check{A} \hookrightarrow (\mathrm{Jac} \otimes X_*)^W$ dual,

connected component of W-invariants:

$$\check{A} = (\mathrm{Jac} \otimes X_*)^{W,0} = P_1.$$

We want to prove: $\mathcal{P}_1^\vee = (\mathrm{Jac} \otimes X^*)_W \overset{!}{\cong} \check{\mathcal{P}}_0$ ← for dual group $\check{G}$

Switch notations from $\check{G}$ to G,
then we want: $(\mathrm{Jac} \otimes X_*)_W \overset{!}{\cong} \mathcal{P}_0$.

Recall:
$\mathrm{Jac} \otimes X_* = H^1(C, T)^0$ and $\mathcal{P}_0 = H^1(C, \mathcal{J}_0)^0$.

Leray spectral sequence for the covering π:
$\mathcal{F}$ sheaf on $\tilde{C}$ $\rightsquigarrow$ $0 \to H^1(C, \pi_* \mathcal{F}) \to H^1(\tilde{C}, \mathcal{F}) \to H^0(C, R^1\pi_* \mathcal{F})$

Now put $\mathcal{F} = T$.
Kummer sequence $0 \to \mu_n \otimes X_* \to \underbrace{\mathbb{G}_m \otimes X_*}_{= T} \to \underbrace{\mathbb{G}_m \otimes X_*}_{= T} \to 0$

$\rightsquigarrow$ $R^1\pi_*(T)$ has no torsion

$\rightsquigarrow$ $H^1(C, \pi_* T)^0 \cong \mathrm{Jac} \otimes X_*$

Recall: $\mathcal{J}_1 = (\pi_* T)^W \supset \mathcal{J}_0 = (\pi_* T)^{W,0}$

Norm map on sheaf level: $\pi_* T \longrightarrow \mathcal{J}_0 \subset \mathcal{J}_1$

$$a \longmapsto \sum_{w \in W} w \cdot a$$

factors over coinvariants:

$$\pi_* T \to (\pi_* T)_W \underset{\sim}{\longrightarrow} \mathcal{J}_0$$

iso on U
(but usually has a cokernel on $C \setminus U$).

Consider

$$\begin{tikzcd}[column sep=small]
\mathrm{Jac} \otimes X_* = H^1(C, \pi_* T)^0 \arrow[rr, "\mathrm{Nm}"] \arrow[dr] & & H^1(C, J_0)^0 = P_0 \\
& (\mathrm{Jac} \otimes X_*)_W \arrow[ur, "\boxed{\text{iso ?!}}"'] &
\end{tikzcd}$$

Clear: Isogeny prime-to-p (separable).

Must prove: $(\mathrm{Jac} \otimes X_*)_W [n] \longrightarrow P_0[n]$ injective for n prime to p,

equivalently: ——— // ——— surjective — // —

$$(\mathrm{Jac} \otimes X_*)_W [n] = H^1(\tilde{C}, \mu_n \otimes X_*)$$

$$= H^1(C, \pi_* (\mu_n \otimes X_*)),$$

$$P_0[n] \hookrightarrow H^1(C, J_0[n]).$$

On sheaf level:

$$\begin{tikzcd}
\pi_*(\mu_n \otimes X_*) \arrow[r] \arrow[rr, bend right, "\eta"'] & \pi_*(\mu_n \otimes X_*)_W \arrow[r] & J_0[n]
\end{tikzcd}$$

$$0 \longrightarrow \ker(\eta) \longrightarrow \pi_*(\mu_n \otimes X_*) \longrightarrow \mathrm{im}(\eta) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{im}(\eta) \longrightarrow J_0[n] \longrightarrow \mathrm{cok}(\eta) \longrightarrow 0$$

$$\begin{tikzcd}
 & H^0(C, \mathrm{cok}(\eta)) \arrow[d] \arrow[dr, "\beta"] & & \\
H^1(C, \pi_*(\mu_n \otimes X_*)) \arrow[r] \arrow[dr, "\alpha"'] & H^1(C, \mathrm{im}(\eta)) \arrow[r] \arrow[d] & H^2(C, \ker(\eta)) \arrow[r, "\gamma"] & H^2(C, \pi_*(\mu_n \otimes X_*)) \\
 & H^1(C, J_0[n]) \arrow[d] & & \\
 & 0 = H^1(C, \mathrm{cok}(\eta)) & &
\end{tikzcd}$$

Want: α surjective

$$\underset{\left(\substack{\text{diagram}\\ \text{chase}}\right)}{\Longleftrightarrow} \beta \text{ surjective onto } \ker(\gamma)$$

Note: $H^2(C, \ker(\eta)) = H^2(C, j_! j^* \ker(\eta)) = H^2_c(U, \ker(\eta))$

because $\ker(\eta) / j_! j^* \ker(\eta)$ is a skyscraper.

Similarly $\ker(\gamma) = \ker\left(H^2_c(U, \ker(\eta)) \to H^2_c(U, \pi_*(\mu_n \otimes X_*))\right)$

Poincaré duality:

$$(\ker(\gamma))^\vee = \mathrm{cok}\left(H^0(U, \pi_*(\mathbb{Z}/n\mathbb{Z} \otimes X^*)) \longrightarrow H^0(U, \ker(\eta)^\vee)\right)$$

Over U, global sections of a local system are the invariants under the covering group W!

$$\Rightarrow (\ker(\gamma))^\vee = \mathrm{cok}\left((\mathbb{Z}/n\mathbb{Z}[W] \otimes X^*)^W \longrightarrow \underset{\substack{\| \\ \mathrm{Hom}(W, \mathbb{Z}/n\mathbb{Z} \otimes X^*)}}{(\ker(\eta)^\vee)^W} \right).$$

Have

$$0 \to \underset{x \in \tilde{U}}{\ker(\eta)_x} \longrightarrow \mu_n[W] \otimes X_* \longrightarrow (\mu_n[W] \otimes X_*)_W \longrightarrow 0,$$

dually

$$0 \to (\mathbb{Z}/n\mathbb{Z}[W] \otimes X^*)^W \longrightarrow \mathbb{Z}/n\mathbb{Z}[W] \otimes X^* \longrightarrow \ker(\eta)^\vee_x \to 0$$

$$\| $$

$$\mathbb{Z}/n\mathbb{Z} \otimes X^*$$

invariants with respect to "geometric" W-action (ie. no action on fibres upstairs on $\tilde{C}$)

$$\Rightarrow (\ker(\eta)^\vee)^W \twoheadrightarrow H^1(W, \mathbb{Z}/n\mathbb{Z} \otimes X^*)$$

Analyze $\operatorname{coker}(\eta)$:

$x \in C^\alpha$

$$\pi_*(\mu_n \otimes X_*)_x = \prod_{w \in W/\langle 1, s_\alpha\rangle} (\mu_n \otimes X_*) \xrightarrow{Nm} J_0[n]$$

$$\underset{\substack{\uparrow \\ w \in W/\langle 1, s_\alpha\rangle}}{(1, \dots, 1, y)} \longmapsto (1 + s_\alpha)(y) = 2y - \check{\alpha} \cdot \langle y, \alpha\rangle$$

$$s_\alpha(y) = y - \alpha^\vee \cdot \underset{\substack{\cap \\ \mu_n}}{\langle y, \alpha\rangle}$$

$$J_0[n]_x = \mu_n \otimes X_*^{\ker(\alpha)}$$

For $n = 2$, $\operatorname{coker}(\eta) = \prod_{\substack{\alpha \in \Phi/W \\ x \in C^\alpha}} (\mu_2 \otimes X_*^{\ker(\alpha)}) / \langle \check{\alpha} \rangle \cdot m_\alpha$